1/ # Math 112 : Introductory Real Analysis

• Integration and differentiation

Thm Let f be integrable on $[a,b]$.
For $a \leq x \leq b$, put $F(x) = \int_a^x f(t)\,dt$. Then F is continuous on $[a,b]$.
Moreover, if f is continuous at $x_0 \in [a,b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

proof) Since f is integrable, f is bounded. Suppose $|f(t)| \leq M$ for $a \leq t \leq b$.

If $a \leq x < y \leq b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t)\,dt \right| \leq M(y-x).$$

Hence, given $\varepsilon > 0$, we see that

$$|F(y) - F(x)| < \varepsilon \quad \text{provided that } |y - x| < \frac{\varepsilon}{M},$$

and therefore F is (uniformly) continuous.

Now suppose that f is continuous at x_0. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon \quad \text{if } |t - x_0| < \delta \text{ and } a \leq t \leq b.$$

Hence, if $x_0 - \delta < s < t < x_0 + \delta$ and $a \leq s < t \leq b$, we have

$$\left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| = \left| \frac{1}{t-s} \int_s^t \big(f(u) - f(x_0)\big)\,du \right|$$

$$\leq \frac{1}{t-s} \int_s^t |f(u) - f(x_0)|\,du < \varepsilon.$$

It follows that $F'(x_0) = f(x_0)$. ∎

<u>Thm</u> (Fundamental theorem of calcalus)

Suppose f is integrable on $[a,b]$. If there is a differentiable function F on $[a,b]$ such that $F' = f$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

proof) Let $\varepsilon > 0$ be given. Choose a partition $P = \{x_0, \cdots, x_n\}$ of $[a,b]$ so that $U(P,f) - L(P,f) < \varepsilon$.

By the mean value theorem, for each $1 \leq i \leq n$, there is $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)\,\Delta x_i = f(t_i)\,\Delta x_i.$$

Thus $\sum_{i=1}^{n} f(t_i)\,\Delta x_i = F(b) - F(a)$, and it follows that

$$\left| F(b) - F(a) - \int_a^b f(x)\,dx \right| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the proof is complete. ■

<u>Thm</u> (Integration by parts)

Suppose F and G are differentiable functions on $[a,b]$ and $f := F'$ and $g := G'$ are integrable. Then

$$\int_a^b F(x)g(x)\,dx = F(b)G(b) - F(a)G(b) - \int_a^b f(x)G(x)\,dx.$$

proof) Apply the previous theorem for $H(x) = F(x)G(x)$ and its derivative. ■

Ex Since $\left(\frac{1}{n+1} x^{n+1}\right)' = x^n$,

$$\int_a^b x^n \, dx = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right).$$

Ex $(e^x)' = e^x$, so $\displaystyle\int_a^b e^x \, dx = e^b - e^a$.

Ex

$$\begin{aligned}\int_a^b x e^x \, dx &= be^b - ae^a - \int_a^b e^x \, dx \\ &= be^b - ae^a - e^b + e^a.\end{aligned}$$